From coin-tossing to rock-paper-scissors and beyond: A log-exp gap theorem for selecting a leader

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Abstract

A class of games for finding a leader among a group of candidates is studied in detail. This class covers games based on coin-tossing and rock-paper-scissors as special cases and its complexity exhibits similar stochastic behaviors: either of logarithmic mean and bounded variance or of exponential mean and exponential variance. Many applications are also discussed.

1 Introduction

Selecting a leader or a representative by fair random mechanisms without relying on *a priori* information of the candidates has long been used in diverse contexts and civilizations. Typical examples range from sortition (or allotment) in the West and rock-paper-scissors (or Janken) in the East; see Wikipedia's pages on sortition and rock-paper-scissors for more information. This paper is concerned with the analysis of a class of *leader selection algorithms* (or *leader election algorithms*) that are used to select a leader among a group of n candidates. These algorithms have widespread applications in diverse areas; see Figure 1 for a summary and below for more descriptions.

Key words: Leader selection, Janken game, recurrence relation, functional equation, Mellin transform, saddle-point method, limit theorems, periodic function, gap theorem, ties.
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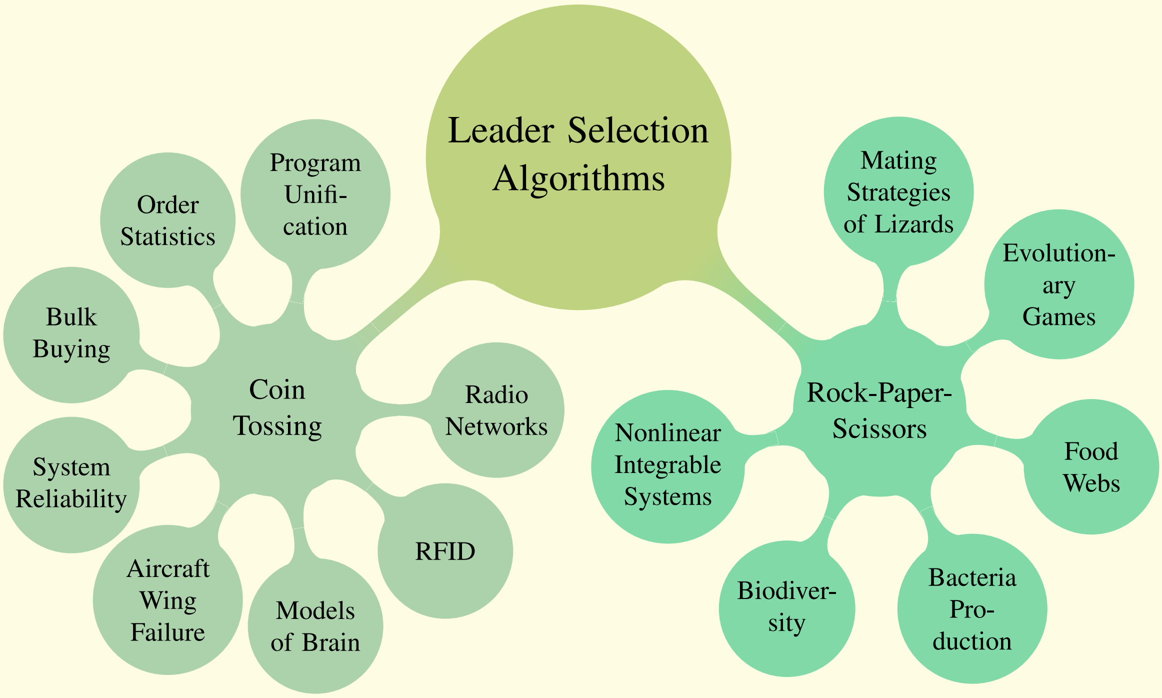


Figure 1: Leader selection procedures and applications.

One easy and efficient way to solve the leader selection problem is to use coin-tossing. A simple such procedure is described as follows. Assume that we have a (possibly biased) coin with two outcomes "head" and "tail". Each of the n candidates tosses an independent coin and those who toss head go on to the next round. In case nobody tosses head, the round is repeated with the same candidates. The procedure ends when only one candidate is left who is then declared the leader. We refer to this simple scheme as **CTLS** (Coin-Tossing Leader Selection). It has been used in many applications, for instance, in the CTM Tree Protocol (after Capetanakis, Tsybakov and Mikhailov), which is used to determine the order in which n processors sharing a common communication channel send their messages; see [4] and [50]. More recent applications are in ad-hoc radio networks (see Chapter 9 of [36]) and RFID systems (see, e.g., [19]).

Many variants of **CTLS** have been proposed and extensively studied. One such variant consists of waiting until everyone is eliminated and then declaring those who stayed the longest in the game as the leaders. This variant, referred to as **MGLS** (Maximum Geometric Leader Selection), amounts to finding the maximum of n i.i.d. geometric distributed random variables. The latter problem has a longer history than the study of **CTLS**, and the earliest publication we found dated back to the early 1950s. Closely connected applications include mathematical models of the brain [37], study of aircraft wing fatigue failure [32] and system reliability in general [51], order statistics of geometric random variables [34], skip-lists [6], bulk buying of possibly defective items [28, 44] and program unification techniques in concurrency enhancement methods [40, 41]; see also the more recent study [7] and the book on bioinformatics [8] for further applications.

The complexity of **CTLS** seemed to have been first analyzed by Bruss and O'Cinneide [3] in the early 1990s, where they attributed the problem of analyzing the number of rounds to identify a leader to Bajaj and Mendieta [1]. In fact, in [3], the authors relied on the above connection to geometric random variables as an approximation to **CTLS**, and gave an analysis of **MGLS** as well. A more detailed analysis of **CTLS** was given independently by Prodinger [39]. In particular, he showed that the expected number of rounds, denoted by $\mathbb{E}(X_n)$, used by **CTLS** to identify a leader among n candidates satisfies

$$\mathbb{E}(X_n) = \log_2 n + \tfrac{1}{2} + P_0(\log_2 n) + O\left(n^{-1}\right), \tag{1}$$

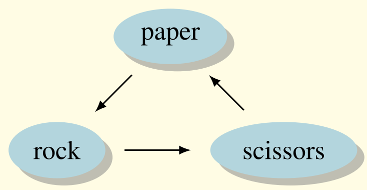
when all coins tossed are unbiased. Here $P_0(u)$ is a bounded periodic function of period 1 with amplitude less than 1.93×10^{-5}. Such a minute yet nonzero periodic oscillation is a characteristic feature for problems of a similar type, and tools for deriving the corresponding Fourier expansions have been the major focus of many papers. A similar periodic phenomena is also present in the variance $\mathbb{V}(X_n)$, which is asymptotic to another periodic function and thus of constant order (again in the unbiased case). This together with the following limit distribution result

$$\mathbb{P}(X_n \leqslant \lfloor \log_2 n \rfloor + \ell) = \frac{2^{\{\log_2 n\}-\ell}}{\exp(2^{\{\log_2 n\}-\ell}) - 1} + O\left(n^{-1}\right), \tag{2}$$

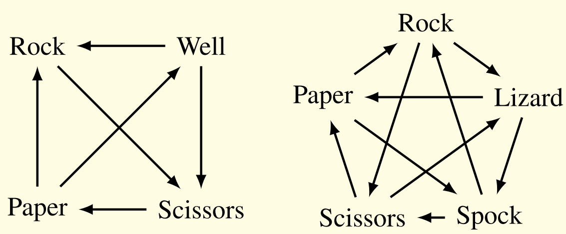
where $\{x\}$ denotes the fractional part of x, was derived in [10]. Moreover, extensions of the above results to biased coins were also considered; see [25].

The logarithmic order (1) shows that **CTLS** is not only simple but also very efficient in terms of cost complexity. We will establish a more general asymptotic pattern of this type and clarify when a leader selection procedure is more efficient than the others.

In addition to coin-tossing with binary outcomes, the natural idea of allowing m-ary outcomes ($m \geqslant 3$) with cyclic dominance has also proved fruitful in diverse applications including leader selection. The simplest such procedure is the "rock-paper-scissors game" (RPS), which is popular in many countries, notably in Japan, where it is called "Janken game", meaning the play between two fists. The rules underlying RPS can be visualized in the digraph to the right, which indicates the dominance relations Many annual tournaments and championships of such games on regional and international level are held, and have received widespread media attention. RPS originated in China and was imported to Japan about a thousand years ago (with different rules). It was modified to the current form of RPS about a century ago, and its use spread later across Asia and to the western and the whole world. See [31, 30, 38] for a detailed account.



Many variants of RPS exist. Examples include the rock-paper-scissors-well game (popular in Germany), and the rock-paper-scissors-Spock-lizard game; see the figure below and Wikipedia's page on RPS for more information.



The usefulness of the Janken game and its variants is not limited to select a winner or a loser, but also to broad applications in other areas. Examples are found in evolutionary game theory (see [18, 43]) and in biology (the field data on alternative male strategies of side-blotched-lizards [42] being a well-known example). Other biological uses are found in food-webs, antibiotic production of bacteria, and biodiversity. In physical applications, Janken games were also encountered in interacting particle systems with cyclic dominance [21] which have a Lotka-Volterra system as a deterministic approximation. These systems can be extended to nonlinear integrable systems; see [2, 22, 23]. The introduction of the spatial structure as lattice Lotka-Volterra system largely enriches the dynamics of interacting particles systems and yields interesting simulation results; see [14, 29, 48, 49].

When used to select a leader among n candidates, it turns out that RPS is very inefficient in that it requires an *exponential* number of rounds to resolve the overwhelming ties, in contrast to the *logarithmic* complexity (1) for **CTLS**. More precisely, the procedure follows along the natural way and if only two different hands are present, then the group whose hand dominates that of the other will go on to the next stage; otherwise, the game is in a tie and has to be repeated. The game ends when only one candidate is left who is then the leader. We will call this procedure **RPSLS**.

Maehara and Ueda [33] proved that when $p_1 = p_2 = p_3 = \frac{1}{3}$, the number of rounds, say Y_n, used by **RPSLS** satisfies $\mathbb{E}(Y_n) \sim \frac{1}{3}\left(\frac{3}{2}\right)^n$ (cf. (1)), and, furthermore,

$$\frac{Y_n}{\frac{1}{3}\left(\frac{3}{2}\right)^n} \xrightarrow{d} \mathrm{Exp}(1),$$

where $\xrightarrow{d}$ denotes convergence in distribution and Exp(1) represents an exponential random variable with mean 1. While the expected exponential complexity of **RPSLS** under uniform distribution of the hands cannot compete with the expected logarithmic one of **CTLS**, for very small n, we have the following numerical values:

Scheme	n	2	3	4	5	6	7	8
CTLS	$\mathbb{E}(X_n)$	4	4.83	5.52	6.09	6.58	6.99	7.35
RPSLS	$\mathbb{E}(Y_n)$	1.5	2.25	3.21	4.49	6.22	8.65	12.1

and we see that in terms of expected number of rounds **RPSLS** is more efficient than **CTLS** when $n \leqslant 6$.

Extensions of **RPSLS** to more hands than three were also studied in distributed computing contexts by Suzaki and Osaki in several papers (see [45, 46, 47]), where partial probabilistic analyses are provided; see also Section 3.

Our aim in this paper is to propose a general model containing all the schemes above as special cases. More precisely, we will define a wide class of generalized Janken games with $m \geqslant 2$ hands, and we will show that the different behaviors mentioned above for **CTLS** and **RPSLS** are prototypical and find their extensive form in a general setting. More precisely, we will show that the complexity of leader selection based on general Janken schemes exhibits a gap theorem: the average number of rounds needed to select a leader is either of *logarithmic* order or of *exponential* order. Moreover, we will establish stronger results including bounded variance and the oscillations of the asymptotic distribution in the log-case, and an exponential limit law in the exp-case.

This paper is structured as follows. In the next section, we define our generalized Janken games and provide general tools for analyzing the number of rounds until a leader is selected. In Section 3, we apply our results to some games. Finally, in Section 4, we discuss extensions and other gap theorems in our model. We conclude the paper with a remark on infinite-hand games.

2 Dichotomous behavior of the complexity

We define first the class of generalized Janken games we will analyze. Then we distinguish between two subclasses of games (log-games and exp-games) for which the cost complexity differs significantly from being logarithmic to being exponential. We will also derive more precise asymptotic approximations.

The analysis of log-games will be more subtle due to the inherent periodic fluctuations in the asymptotic approximations to the moments and the limit law, which will be clarified by generalizing our recent approach from [15]. The analysis of exp-games, on the other hand, is more straightforward and our results will follow by the method of moments.

2.1 Generalized Janken games

Let $m \geqslant 2$, and we are given m hands $\{\mathscr{H}_1, \dots, \mathscr{H}_m\}$ with (positive) probabilities $\{p_1, \dots, p_m\}$. A generalized Janken game is played as follows. Assume that the set of hands the n players may choose equals $S = \{\mathscr{H}_{i_1}, \dots, \mathscr{H}_{i_\ell}\}$. Then there are two situations:

- S is a (clear-cut) *win-or-defeat (abbreviated as WOD) set*, i.e., $S = W \cup D$ with $W \cap D = \emptyset$, where $W = \{\mathscr{H}_{j_1}, \dots, \mathscr{H}_{j_r}\}$, $1 \leqslant r < \ell$ for the current choice of hands S is the set of winning hands, meaning that players having chosen these hands continue to play in the next round, and $D = \{\mathscr{H}_{k_1}, \dots, \mathscr{H}_{k_{\ell-r}}\}$ is the set of losing hands, meaning that players having chosen these hands are eliminated after the current round.
- The hands in S result in a *tie*, meaning that no one is eliminated and the round is non-conclusive and has to be repeated.

The generalized Janken game is played one round after another until a single player remains who is then the leader. This is always possible if there is at least one WOD set whose cardinality is two. We refer to these games as **`GJLS`**.

For analysis purposes, we introduce more notations. First, given $S = \{\mathscr{H}_{i_1}, \dots, \mathscr{H}_{i_\ell}\}$, define

$$\pi_n^{(S)} := \sum_{\substack{j_1+\cdots+j_\ell=n \\ j_1,\dots,j_\ell \geqslant 1}} \binom{n}{j_1, \dots, j_\ell} p_{i_1}^{j_1} \cdots p_{i_\ell}^{j_\ell}.$$

Then, by the inclusion-exclusion principle,

$$\begin{aligned}\pi_n^{(S)} &= \left(p_{i_1} + \cdots + p_{i_\ell}\right)^n - \left(p_{i_2} + \cdots + p_{i_\ell}\right)^n - \cdots - \left(p_{i_1} + \cdots + p_{i_{\ell-1}}\right)^n \\ &\quad + \left(p_{i_3} + \cdots + p_{i_\ell}\right)^n + \left(p_{i_2} + p_{i_4} + \cdots + p_{i_\ell}\right)^n + \cdots + \left(p_{i_1} + \cdots + p_{i_{\ell-2}}\right)^n \pm \cdots .\end{aligned}$$

From this, we see that, for large n,

$$\pi_n^{(S)} \sim \left(p_{i_1} + \cdots + p_{i_\ell}\right)^n . \tag{3}$$

Let $\mathscr{W}\mathscr{D}$ denote the set of all WOD sets. Define now two game indices:

$$\rho := \max_{\{\mathscr{H}_{i_1},\dots,\mathscr{H}_{i_\ell}\} \in \mathscr{W}\mathscr{D}} \{p_{i_1} + \cdots + p_{i_\ell}\}, \tag{4}$$

and ν the number of WOD sets attaining the maximum value ρ. We distinguish between two cases:

- log-game: $\rho = 1$;
- exp-game: $\rho < 1$.

Note that a log-game occurs if and only if $\{\mathscr{H}_1, \ldots, \mathscr{H}_m\}$ is itself a WOD set, and in this case $\nu = 1$. Also log-games are more meaningful when the hands are generated by purely random mechanisms but not by intentional calculation (by which the outcomes would become deterministic and uninteresting).

We are interested in the number of rounds X_n used by n players to select a leader by **GJLS**, which is one of the most important cost measures of the game. This number satisfies, by considering the size of the winning group after the first round of **GJLS**, the following distributional recurrence

$$X_n \stackrel{d}{=} X_{I_n} + 1 \qquad (n \geqslant 2), \tag{5}$$

where the X_n's and I_n's are independent, $X_1 = 0$, and for $1 \leqslant j \leqslant n$,

$$\mathbb{P}(I_n = j) = \begin{cases} 1 - \varpi_n, & \text{if } j = n; \\ \displaystyle\sum_{\substack{S \in \mathscr{W}\mathscr{D} \\ S = W \cup D}} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)}, & \text{if } 1 \leqslant j < n, \end{cases} \tag{6}$$

where $\varpi_n := \sum_{S \in \mathscr{W}\mathscr{D}} \pi_n^{(S)}$ stands for the probability of no tie occurring. From (3) and by the definitions of ρ and ν, we have

$$\varpi_n \sim \nu \rho^n. \tag{7}$$

See also [17, 27] for recurrences similar to (5).

Alternatively, instead of considering only one round, one may also wait for a random number of times T_n until a WOD set is reached. This then yields the following alternative distributional recurrence for X_n

$$X_n \stackrel{d}{=} X_{J_n} + T_n \qquad (n \geqslant 2), \tag{8}$$

where the X_n's, J_n's, and T_n's are independent, T_n is a geometric distributed random variable with parameter ϖ_n, $X_1 = 0$ and for $1 \leqslant j < n$,

$$\mathbb{P}(J_n = j) = \frac{1}{\varpi_n} \sum_{\substack{S \in \mathscr{W}\mathscr{D} \\ S = W \cup D}} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)}. \tag{9}$$

Both forms of the distributional recurrence will be useful for us; (5) will be used in the analysis of log-games, whereas (8) is advantageous in the analysis of exp-games.

2.2 Log-games

In this subsection, we consider log-games for which $\rho = 1$ and $\nu = 1$. In this case the whole set of hands $\{\mathscr{H}_1, \ldots, \mathscr{H}_m\}$ is itself a WOD set, and we define

$$\alpha := \sum_{\mathscr{H}_j \text{ is a winning hand in } \{\mathscr{H}_1, \ldots, \mathscr{H}_m\}} p_j.$$

Denote by X_n the number of rounds to select a leader by **GJLS**. Such games are marked by its complexity X_n satisfying logarithmic mean, bounded variance and periodic oscillations of the asymptotic distributions. This is the same pattern as for **CTLS**; see Section 1. For the proofs, we extend the analytic approach used in our previous paper [15].

Mean value. Let $\mu_n := \mathbb{E}(X_n)$. Then, by (5), we obtain that

$$\mu_n = \sum_{1\leqslant j<n} \binom{n}{j} \mu_j \sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D}} \pi_j^{(W)} \pi_{n-j}^{(D)} + (1-\varpi_n)\mu_n + 1,$$

for $n \geqslant 2$, with the initial condition $\mu_1 = 0$. To solve this recurrence, we consider first the Poisson generating function $\tilde{f}_1(z) := e^{-z}\sum_{n\geqslant 1} \mu_n \frac{z^n}{n!}$.

In what follows, an "exponentially small term" is used to mean an entire function that is bounded above by $e^{-c\Re(z)}$ for some $c > 0$ as $|z| \to \infty$ in the half-plane $\Re(z) > 0$.

Lemma 1. *The Poisson generating function of μ_n satisfies the functional equation*

$$\tilde{f}_1(z) = \tilde{f}_1(\alpha z) + 1 + \sum_{1\leqslant j\leqslant k} \lambda_j e^{-\beta_j z} \tilde{f}_1(\alpha_j z) - (1+z)e^{-z}, \tag{10}$$

with $\tilde{f}_1(0) = 0$, where $\lambda_j \in \{-1, 1\}$ and $0 < \alpha_j, \beta_j < 1$ are constants.

Proof. Since $\rho = 1$, we see that $1 - \varpi_n$ consists only of exponentially small terms. This implies that we can arrange the terms and write

$$e^{-z}\sum_{n\geqslant 1}(1-\varpi_n)\mathbb{E}(X_n)\frac{z^n}{n!} = \sum_{1\leqslant j\leqslant k} \lambda_j e^{-\beta_j z}\tilde{f}_1(\alpha_j z).$$

On the other hand

$$\sum_{n\geqslant 2}\sum_{1\leqslant j<n}\sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D}} \binom{n}{j}\pi_j^{(W)}\pi_{n-j}^{(D)}\mathbb{E}(X_j)\frac{z^n}{n!} = \sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D}} \left(\sum_{n\geqslant 1}\pi_n^{(D)}\frac{z^n}{n!}\right)\left(\sum_{n\geqslant 1}\pi_n^{(W)}\mathbb{E}(X_n)\frac{z^n}{n!}\right).$$

The largest terms (as $|z| \to \infty$ in $\Re(z) > 0$) comes from the whole set $S = \{\mathscr{H}_1, \dots, \mathscr{H}_m\}$, which is itself a WOD set and produces terms of the form (by (3))

$$e^{-z}\left(\sum_{n\geqslant 1}\pi_n^{(D)}\frac{z^n}{n!}\right)\left(\sum_{n\geqslant 1}\pi_n^{(W)}\mathbb{E}(X_n)\frac{z^n}{n!}\right) = \tilde{f}_1(\alpha z) + \sum_{1\leqslant j\leqslant k}\lambda_j e^{-\beta_j z}\tilde{f}_1(\alpha_j z),$$

for some $\lambda_j \in \{-1, 1\}$ and $0 < \alpha_j, \beta_j < 1$ (whose values may differ from one occurrence to another). For all other WOD sets different from $\{\mathscr{H}_1, \dots, \mathscr{H}_m\}$, we have $\sum_{\mathscr{H}_i\in S} p_i < 1$ and they can be regrouped as

$$e^{-z}\sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D\\ S\neq\{\mathscr{H}_1,\dots,\mathscr{H}_m\}}} \left(\sum_{n\geqslant 1}\pi_n^{(D)}\frac{z^n}{n!}\right)\left(\sum_{n\geqslant 1}\pi_n^{(W)}\mathbb{E}(X_n)\frac{z^n}{n!}\right) = \sum_{1\leqslant j\leqslant k}\lambda_j e^{-\beta_j z}\tilde{f}_1(\alpha_j z),$$

where $\lambda_j \in \{-1, 1\}$ and $0 < \alpha_j, \beta_j < 1$. The remaining computations are straightforward. Thus, our claim is proved. ∎

We will write (10) as

$$\tilde{f}_1(z) = \tilde{f}_1(\alpha z) + 1 + \tilde{\phi}_1(z), \tag{11}$$

where $\tilde{\phi}_1(z)$ is exponentially small in the half-plane $\Re(z) > 0$.

Variance. Consider now the Poisson generating function of $\mathbb{E}(X_n^2)$, denoted by $\tilde{f}_2(z) := e^{-z}\sum_{n\geqslant 1}\mathbb{E}(X_n^2)\frac{z^n}{n!}$. Define $\tilde{V}(z) := \tilde{f}_2(z) - \tilde{f}_1(z)^2$.

Lemma 2. *The function $\tilde{V}(z)$ satisfies the functional equation*

$$\tilde{V}(z) = \tilde{V}(\alpha z) + \tilde{\phi}_3(z), \tag{12}$$

where $\tilde{\phi}_3(z)$ is an exponentially small term.

Proof. A similar analysis as that used for $\tilde{f}_1(z)$ leads to the functional equation

$$\tilde{f}_2(z) = \tilde{f}_2(\alpha z) + 2\tilde{f}_1(\alpha z) + 1 + \tilde{\phi}_2(z), \tag{13}$$

where $\tilde{\phi}_2(z)$ consists of exponentially small terms (involving both $\tilde{f}_1(z)$ and $\tilde{f}_2(z)$). Then (12) follows from (11) and (13). ∎

Asymptotics and JS-admissibility. From the Poisson generating functions $\tilde{f}_1(z)$ (which is indeed the expected cost X_n when n itself follows a Poisson(z) distribution), we can recover the asymptotic behaviors of $\mathbb{E}(X_n)$ by the relation

$$\mu_n = n![z^n]e^z\tilde{f}_1(z).$$

This can be done by several means, and a by-now standard approach are the so-called analytic de-Poissonization techniques largely developed by Jacquet and Szpankowski in [26], which rely on the saddle-point method (see [13]). The use of such an approach can be further schematized by introducing the notion of $\mathscr{J}\mathscr{S}$-admissible functions, which we formulated in [20]. We briefly sketch the underlying ideas, and refer the interested readers to our previous papers for more details [15, 20]. Similarly, since μ_n is of logarithmic order and the variance is bounded, the function $\tilde{V}(z)$ will provide a sufficiently good approximation to the variance $\mathbb{V}(X_n)$.

We recall the following definition from [20]. Let

$$\mathscr{C}_\varepsilon := \{z : |\arg(z)| \leqslant \varepsilon\},$$

where $\varepsilon > 0$ is an arbitrary constant (which will subsequently be used as a generic symbol whose value may change from one occurrence to another).

Definition 1. Let $\tilde{f}(z)$ be an entire function and $\xi, \eta \in \mathbb{R}$. Then $\tilde{f}(z)$ is *JS-admissible*, written $\tilde{f} \in \mathscr{J}\mathscr{S}$ (or more precisely, $\tilde{f} \in \mathscr{J}\mathscr{S}_{\xi,\eta}$), if for $0 < \phi < \pi/2$ and all $|z| \geqslant 1$ the following two conditions hold.

(I) Uniformly for $z \in \mathscr{C}_\varepsilon$, $\tilde{f}(z) = O\big(|z|^\xi(\log_+|z|)^\eta\big)$, where $\log_+ x := \log(1+x)$.

(O) Uniformly for $\phi \leqslant |\arg(z)| \leqslant \pi$, $f(z) := e^z\tilde{f}(z) = O\big(e^{(1-\varepsilon)|z|}\big)$, for some $\varepsilon > 0$.

Such functions enjoy closure properties under several different operations, and it is these properties that make such a notion really useful. Also if $f \in \mathscr{J}\mathscr{S}_{\xi,\eta}$, then

$$n![z^n]f(z) = \sum_{0\leqslant j<2k}\frac{\tilde{f}^{(j)}(n)}{j!}\tau_j(n) + O\big(n^{\xi-k}(\log n)^\eta\big) \qquad (k \geqslant 1),$$

where for $j \geqslant 1$ $\tau_j(n) := \sum_{0\leqslant \ell\leqslant j}\binom{j}{\ell}(-1)^{j-\ell}\frac{n!n^{j-\ell}}{(n-\ell)!}$ are Charlier polynomials. In particular,

$$n![z^n]f(z) = \tilde{f}(n) - \frac{\tilde{f}''(n)}{2}n + \frac{\tilde{f}'''(n)}{3}n + O\big(n^{\xi-2}(\log n)^{\eta}\big). \tag{14}$$

Specific to our analysis, we need additionally the following property.

Lemma 3. *Let $\tilde{f}$ and $\tilde{g}$ be two entire functions satisfying the functional equation*

$$\tilde{f}(z) = \tilde{f}(\alpha z) + \sum_{1\leqslant j\leqslant k}\lambda_j e^{-\beta_j z}\tilde{f}(\alpha_j z) + \tilde{g}(z), \tag{15}$$

with $\tilde{f}(0) = \tilde{g}(0) = 0$, where $\lambda_j \in \mathbb{R}$, $\beta_j > 0$ and $\alpha, \alpha_j \in (0,1)$. Then

$$\tilde{g} \in \mathscr{JS} \quad \Longleftrightarrow \quad \tilde{f} \in \mathscr{JS}.$$

The proof is similar to that of Proposition 3 in the Appendix of [15], and is omitted here.

From this and (11), (13), we see that $\tilde{f}_1 \in \mathscr{JS}_{0,1}$ and $\tilde{f}_2 \in \mathscr{JS}_{0,2}$. By (14) and the arguments used for the variance in [20], we obtain that

$$\mathbb{E}(X_n) = \tilde{f}_1(n) + O\left(n^{-1}\right), \qquad \mathbb{V}(X_n) = \tilde{V}(n) + O\left(n^{-1}\right). \tag{16}$$

Thus for asymptotic purposes, we can entirely focus on the Poisson model. A standard approach to the asymptotics of the Poisson generating function for large $|z|$ in similar situations is based on the Mellin transform techniques

$$F^*(s) = \mathscr{M}[\tilde{f}; s] := \int_0^\infty \tilde{f}(z)z^{s-1}\mathrm{d}z;$$

see [11] for an authoritative survey. We first derive the asymptotic behavior for functions satisfying the more general equation (15). Let $L := \log(1/\alpha)$, $\chi_k := 2k\pi i/L$ and

$$\tilde{\phi}(z) := \sum_{1\leqslant j\leqslant k}\lambda_j e^{-\beta_j z}\tilde{f}(\alpha_j z).$$

Proposition 1. *(i) Assume that $\tilde{g} \in \mathscr{JS}_{\xi,\eta}$ with $\xi < 0$. Then, as $|z| \to \infty$ in the sector $\mathscr{C}_\varepsilon$,*

$$\tilde{f}(z) = \frac{1}{L}\sum_{k\in\mathbb{Z}}\big(\Phi^*(\chi_k) + G^*(\chi_k)\big)z^{-\chi_k} + O\big(|z|^{-\min\{1,\xi\}}(\log|z|)^{\eta}\big), \tag{17}$$

where $\Phi^*(s) := \mathscr{M}[\tilde{\phi}; s]$ and $G^*(s) := \mathscr{M}[\tilde{g}; s]$.

(ii) Assume that $\tilde{g} \in \mathscr{JS}$ and $\tilde{g}(z) = c + O\left(|z|^{-\xi}\right)$ as $|z| \to \infty$ in the sector $\mathscr{C}_\varepsilon$. Then

$$\tilde{f}(z) = c\log_{1/\alpha} z + \frac{c}{2} + \frac{d+\Phi^*(0)}{L} + \sum_{k\neq 0}\big(\Phi^*(\chi_k) + G^*(\chi_k)\big)z^{-\chi_k} + O\big(|z|^{-\min\{1,\xi\}}\big),$$

as $|z| \to \infty$ in $\mathscr{C}_\varepsilon$, where $d = \lim_{s\to 0}(G^*(s) + c/s)$.

Remark 1. Note that the Fourier coefficients of the periodic functions in the above result are less explicit due to the occurrence of $\tilde{f}(z)$ in $\tilde{\phi}(z)$. However, in some situations (e.g., **CTLS**) they can be made explicit; see [39] and the remarks in Section 3.

Proof. Consider part *(i)*. The assumptions imply that $G^*(s)$ exists in the strip $-1 < \Re(s) < -\xi$ and

$$F^*(s) := \mathscr{M}[\tilde{f}; z] = \frac{\Phi^*(s) + G^*(s)}{1 - \alpha^{-s}}, \qquad \Re(s) \in (-1, 0). \tag{18}$$

Note that from the assumptions, Lemma 3 and the Exponential Smallness Lemma in [16], we obtain that $F^*(s)$ decays exponentially fast along vertical lines in $-1 < \Re(s) < -\xi$. Consequently, by standard Mellin argument, we deduce (17).

For part *(ii)*, by the Direct Mapping Theorem in [11], we see that $G^*(s)$ has a simple pole at $s = 0$ with the singularity expansion $G^*(s) \asymp c/s + d + \cdots$. The rest of the proof then follows as in part *(i)*. ∎

Collecting all results, we then derive the asymptotics of the mean and the variance.

Theorem 1 (Log-games)**.** *If $\rho = 1$, then the number of rounds X_n to find a leader used by* ***GJLS*** *satisfies*

$$\begin{cases} \mathbb{E}(X_n) = \log_{1/\alpha} n + P_1(\log_{1/\alpha} n) + O\left(n^{-1}\right), \\ \mathbb{V}(X_n) = P_2(\log_{1/\alpha} n) + O\left(n^{-1}\right), \end{cases}$$

where $P_1(t)$, $P_2(t)$ are bounded, 1-periodic functions.

Limit law. We now turn to the limit law. We begin with considering the Poisson generating function of the probability generating function of X_n $\tilde{P}(y,z) := e^{-z}\sum_{n\geqslant 1}\mathbb{E}\left(y^{X_n}\right)\frac{z^n}{n!}$, which satisfies the functional equation

$$\tilde{P}(y,z) = y\tilde{P}(y,\alpha z) + y\sum_{1\leqslant j\leqslant k}\lambda_j e^{-\beta_j z}\tilde{P}(y,\alpha_j z) + (1-y)ze^{-z},$$

where $\lambda_j \in \{-1, 1\}$ and $\alpha, \alpha_j, \beta_j \in (0,1)$. Observe that

$$\frac{\tilde{P}(y,z)}{1-y} = \sum_{\ell\geqslant 0}\tilde{A}_\ell(z)y^\ell, \quad \text{where} \quad \tilde{A}_\ell(z) := e^{-z}\sum_{n\geqslant 0}\mathbb{P}(X_n \leqslant \ell)\frac{z^n}{n!}.$$

By dividing the above functional equation by $1-y$ and reading off coefficients, we obtain $\tilde{A}_0(z) = ze^{-z}$ and the recursive functional equation

$$\tilde{A}_{\ell+1}(z) = \tilde{A}_\ell(\alpha z) + \sum_{1\leqslant j\leqslant k}\lambda_j e^{-\beta_j z}\tilde{A}_\ell(\alpha_j z), \qquad (\ell \geqslant 0). \tag{19}$$

The equations here are similar to those in [15], and the remaining analysis follows along the same lines there, which we now sketch. First, define $\tilde{R}_0(z) := \tilde{A}_0(z)$ and for $\ell \geqslant 0$

$$\tilde{R}_{\ell+1}(z) := \sum_{1\leqslant j\leqslant k}\lambda_j e^{-\beta_j z}\tilde{A}_\ell(\alpha_j z).$$

Then

$$\tilde{A}_{\ell+1}(z) = \tilde{A}_\ell(\alpha z) + \tilde{R}_{\ell+1}(z) = \sum_{0\leqslant j\leqslant \ell}\tilde{R}_j\left(\alpha^{\ell-j}z\right).$$

For the asymptotics, we need the JS-admissibility in a stronger uniform form.

Lemma 4. *For $\ell \geqslant 1$, $\tilde{A}_\ell(z)$ is uniformly JS-admissible, i.e., for $|\arg(z)| \leqslant \varepsilon, 0 < \varepsilon < \pi/2$,*

$$\tilde{A}_\ell(z) = O\left(|z|^{\varepsilon_1}\right), \tag{20}$$

uniformly in z, and, for $\varepsilon \leqslant |\arg(z)| \leqslant \pi$,

$$e^z \tilde{A}_\ell(z) = O\big(e^{(1-\varepsilon_2)|z|}\big), \tag{21}$$

uniformly in z. Here the involved constants in both cases are absolute and $\varepsilon_1, \varepsilon_2$ are positive constants.

Proof. The proof is similar to that of Lemma 3 in [15]. ∎

Then, by standard de-Poissonization arguments [20, 26], we deduce that

$$\mathbb{P}(X_n \leqslant \ell) = \sum_{0 \leqslant j \leqslant \ell} \tilde{R}_j\big(\alpha^{\ell-j} n\big) + O\left(n^{-1+\varepsilon}\right),$$

uniformly in ℓ. On the other hand, by the definition of $\tilde{R}_j(x)$, we see that $|\tilde{R}_j(x)| \leqslant e^{-px}$ for some $p > 0$. Consequently,

$$\sum_{j > \ell} \tilde{R}_j\big(\alpha^{\ell-j} n\big) \leqslant \sum_{j > \ell} e^{-p\alpha^{\ell-j} n} = \sum_{j \geqslant 1} e^{-p\alpha^{-j} n} = O\left(e^{-pn}\right).$$

The latter implies that

$$\mathbb{P}(X_n \leqslant \ell) = \sum_{j \geqslant 0} \tilde{R}_j\big(\alpha^{\ell-j} n\big) + O\left(n^{-1+\varepsilon}\right),$$

uniformly in ℓ. Replacing now ℓ by $\lfloor \log_{1/\alpha} n \rfloor + \ell$, and writing $\vartheta(n) = \{\log_{1/\alpha} n\}$, we obtain

$$\mathbb{P}(X_n \leqslant \lfloor \log_{1/\alpha} n \rfloor + \ell) = \sum_{j \geqslant 0} \tilde{R}_j\big(\alpha^{-\vartheta(n)+\ell-j}\big) + O\left(n^{-1+\varepsilon}\right).$$

We summarize the analysis as follows.

Theorem 2 (Log-games)**.** *If $\rho = 1$, then the number X_n of rounds used by **GJLS** until a leader is selected satisfies*

$$\mathbb{P}(X_n \leqslant \lfloor \log_{1/\alpha} n \rfloor + \ell) = \sum_{j \geqslant 0} \tilde{R}_j\big(\alpha^{-\vartheta(n)+\ell-j}\big) + O\left(n^{-1}\right),$$

uniformly in ℓ.

The improved error term comes from refining the above analysis (by expanding more terms in the de-Poissonization procedure).

Remark 2. It is possible to derive more explicit expressions for $\tilde{A}_\ell(z)$ and $\tilde{R}_j(z)$ in Theorem 2, but they are generally messy. For that purpose, rewrite (19) as

$$\tilde{A}_{\ell+1}(z) = \sum_{0 \leqslant j \leqslant k} \lambda_j e^{-\beta_j z} \tilde{A}_\ell(\alpha_j z),$$

where $\lambda_0 = 1$, $\alpha_0 = \alpha$ and $\beta_0 = 0$. Iterating yields

$$\tilde{A}_\ell(z) = \sum_{\mathbf{j}=(j_1,\dots,j_\ell)\in\{0,\dots,k\}^\ell} \lambda_{\mathbf{j}} e^{-\sum_{i=1}^{\ell} \alpha_{j_1}\cdots\alpha_{j_{i-1}}\beta_{j_i} z} \tilde{A}_0(\alpha_{\mathbf{j}} z),$$

where $\lambda_{\mathbf{j}} = \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_\ell}$, and $\alpha_{\mathbf{j}} = \alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_\ell}$. Since $\tilde{A}_0(z) = ze^{-z}$, we then have

$$\tilde{A}_\ell(z) = z \sum_{\mathbf{j}=(j_1,\dots,j_\ell)\in\{0,\dots,k\}^\ell} \alpha_{\mathbf{j}}\lambda_{\mathbf{j}} e^{-\sum_{i=1}^{\ell+1} \alpha_{j_1}\cdots\alpha_{j_{i-1}}\beta_{j_i} z},$$

with $j_{\ell+1} := 1$. We will give some examples for which this expression simplifies in Section 3.

2.3 Exp-games

In this section we consider the exp-games for which $\rho < 1$ and the probability ϖ_n of no tie occurring satisfies $\varpi_n \sim \nu\rho^n$. We show that X_n converges (with all its moments) to an exponential distribution.

Theorem 3 (Exp-games)**.** *Assume $\rho < 1$. The number of rounds X_n used to select a leader by* ***GJLS*** *converges in distribution and with all moments to an exponential distribution:*

$$\nu\rho^n X_n \xrightarrow{d} \mathrm{Exp}(1),$$

where Exp(1) *denotes an exponential random variable with mean one. In particular,*

$$\mathbb{E}(X_n) \sim \nu^{-1}\rho^{-n}.$$

The proof relies on the recurrence (8) using the method of moments.
First, taking the m-th moment on both sides of (8) gives ($\mu_{n,m} := \mathbb{E}\left(X_n^m\right)$)

$$\begin{aligned}
\mu_{n,m} &= \sum_{0\leqslant k\leqslant m} \binom{m}{k} \mathbb{E}(T_n^{m-k}) \mu_{J_n,k} \\
&= \frac{1}{\varpi_n} \sum_{0\leqslant k\leqslant m} \binom{m}{k} \mathbb{E}(T_n^{m-k}) \sum_{1\leqslant j<n} \sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D}} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)} \mu_{j,k} \qquad (22)\\
&= \frac{1}{\varpi_n} \sum_{1\leqslant j<n} \sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D}} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)} \mu_{j,m} + \kappa_{n,m},
\end{aligned}$$

(by singling out the term with $k = m$), where

$$\kappa_{n,m} := \frac{1}{\varpi_n} \sum_{0\leqslant k<m} \binom{m}{k} \mathbb{E}(T_n^{m-k}) \sum_{1\leqslant j<n} \sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D}} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)} \mu_{j,k}. \qquad (23)$$

The proof then proceeds in two steps: first, we derive an upper bound for $\mu_{n,m}$, and then we refine it and get a more precise asymptotic approximation to $\mu_{n,m}$.

Lemma 5. *For $m \geqslant 0$, $\mathbb{E}(T_n^m) = O\left(\rho^{-nm}\right)$.*

Proof. This follows from the fact that $\mathbb{E}(T_n^m) = P\left(\varpi_n^{-1}\right)$, where $P(x)$ is a polynomial of degree m (without constant term), and the asymptotics (7) of ϖ_n. ∎

We now use this to find a similar bound for $\mu_{n,m}$.

Lemma 6. *For $m \geqslant 0$, $\mu_{n,m} = O\left(\rho^{-nm}\right)$.*

Proof. We prove by induction that

$$\mu_{n,m} \leqslant c_m \rho^{-nm}, \tag{24}$$

for some constants c_m.

Assume that the bound (24) is proved for all moments of order $< m$ and for the m-th moment for all indices $< n$. To prove it for n, we use (23). First, observe that

$$\kappa_{n,m} = O\left(\frac{1}{\varpi_n} \sum_{0 \leqslant k < m} \rho^{-n(m-k)} \sum_{1 \leqslant j < n} \sum_{\substack{S \in \mathscr{W}\mathscr{D} \\ S = W \cup D}} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)} \rho^{-jk}\right) = O\left(\rho^{-nm}\right),$$

implying that $\kappa_{n,m} \leqslant d_m \rho^{-nm}$ for a suitable constant d_m. By this bound, induction hypothesis and (23), we obtain that

$$\begin{aligned} \mu_{n,m} &\leqslant \frac{c_m}{\varpi_n} \sum_{1 \leqslant j < n} \sum_{\substack{S \in \mathscr{W}\mathscr{D} \\ S = W \cup D}} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)} \rho^{-jm} + d_m \rho^{-nm} \\ &\leqslant \left(\rho^m c_m + d_m\right) \rho^{-nm} \leqslant c_m \rho^{-nm}, \end{aligned}$$

where the last step follows by choosing c_m such that $c_m \geqslant d_m/(1-\rho^m)$. ∎

Now we refine Lemma 5.

Lemma 7. *For $m \geqslant 0$,*

$$\mathbb{E}(T_n^m) \sim \frac{m!}{\nu^m \rho^{nm}}. \tag{25}$$

Proof. The moment generating function of $\varpi_n T_n$ is given by

$$\mathbb{E}(e^{\varpi_n T_n t}) = \frac{\varpi_n e^{\varpi_n t}}{1-(1-\varpi_n)e^{\varpi_n t}} \to \frac{1}{1-t},$$

for $|t| < 1$, since $\varpi_n \to 0$. The latter is the moment generating function of an exponential distribution with mean 1. Consequently, we have $\mathbb{E}(\varpi_n T_n)^m \to m!$ for $m \geqslant 0$, which, together with the asymptotics of ϖ_n, proves (25). ∎

We now prove a similar result for $\mu_{n,m}$. Rewrite (22) as $\mathbb{E}(X_n^m) = \mathbb{E}(T_n^m) + \kappa'_{n,m}$, where

$$\kappa'_{n,m} := \frac{1}{\varpi_n} \sum_{1 \leqslant k \leqslant m} \binom{m}{k} \mathbb{E}(T_n^{m-k}) \sum_{1 \leqslant j < n} \sum_{\substack{S \in \mathscr{W}\mathscr{D} \\ S = W \cup D}} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)} \mu_{j,k}. \tag{26}$$

Lemma 8. *For $m \geqslant 0$,*

$$\mu_{n,m} \sim \frac{m!}{\nu^m \rho^{nm}}. \tag{27}$$

Proof. We use (26). First we have

$$\begin{aligned}
\kappa'_{n,m} &= O\Bigg(\rho^{-n} \sum_{1\leqslant k\leqslant m} \rho^{-n(m-k)} \sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D}} \sum_{1\leqslant j<n} \binom{n}{j} \pi_j^{(W)} \pi_{n-j}^{(D)} \rho^{-jk}\Bigg)\\
&= O\Bigg(\sum_{1\leqslant k\leqslant m} \rho^{n(m-k+1)} \sum_{\substack{S\in\mathscr{W}\mathscr{D}\\ S=W\cup D}} \Big(\rho^{-k}\sum_{j\in W} p_j + \sum_{\ell\in D} p_\ell\Big)^n\Bigg)\\
&= O\big((\tilde\rho/\rho)^n \rho^{-nm}\big),
\end{aligned}$$

where $\tilde\rho = \max\left\{\sum_{j\in W} p_j + \rho\sum_{j\in D} p_j \;:\; S = W\cup D\in\mathscr{W}\mathscr{D}\right\}$. Since $\tilde\rho<\rho$, we see that $(\tilde\rho/\rho)^n$ is exponentially smaller than 1. Thus, by substituting the above estimate and Lemma 7 into (26), we prove (27). ∎

The estimates (27) imply the convergence in distribution of X_n by the method of moment. This proves Theorem 3.

3 Applications

We apply in this section the previous theorems to a few concrete cases.

CTLS. For leader selection by coin tossing, the set of hands is given by $\{\mathscr{H}_1, \mathscr{H}_2\}$, where $\mathscr{H}_1$ corresponds to head, and $\mathscr{H}_2$ to tail. The only one WOD set is $\{\mathscr{H}_1, \mathscr{H}_2\}$, where $\mathscr{H}_1$ is the winning hand and $\mathscr{H}_2$ the losing hand. The remaining subsets of hands all lead to ties. Thus **CTLS** is a log-game and both Theorems 1 and 2 apply.

For instance, in the case of an unbiased coin (i.e., $p_1 = p_2 = \frac12$), we obtain $\varpi_n = 1-2^{1-n}$, and for (6)

$$\mathbb{P}(I_n = j) = \begin{cases} 2^{1-n}, & \text{if } j=n;\\ \binom{n}{j}2^{-n}, & \text{if } 1\leqslant j<n.\end{cases}$$

Thus the functional equation (15) satisfied by the Poisson generating functions of the moments has the form

$$\tilde f(z) = \big(1+e^{-z/2}\big)\tilde f\left(\frac z2\right) + \tilde g(z),$$

with $\tilde f(0) = \tilde g(0) = 0$, which can be further simplified by considering $\hat f(z) := \frac{\tilde f(z)}{1-e^{-z}}$ and $\hat g(z) := \frac{\tilde g(z)}{1-e^{-z}}$, so that (by $\hat g(0)=0$)

$$\hat f(z) = \hat f\left(\frac z2\right) + \hat g(z) = \sum_{j\geqslant 0}\hat g\left(\frac{z}{2^j}\right).$$

From this we can derive an explicit expression for the Fourier coefficients in Proposition 1 (compare Remark 1); see [39] for details.

Furthermore, the Poisson generating function for the distribution function satisfies (see (19))

$$\tilde A_{\ell+1}(z) = \big(1+e^{-z/2}\big)\tilde A_\ell\left(\frac z2\right).$$

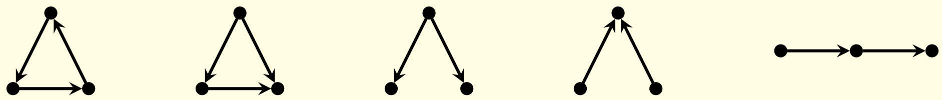
Then by Remark 2, the sum on the right-hand side in Theorem 2 indeed simplifies to the expression in (2).

In addition, our results also apply to the case of an unbiased coin ($p_1 \neq p_2$) and one then recovers the results from [25]. In this case, the sum on the right-hand side in Theorem 2 can be written more elegantly as an integral with respect to a suitable defined point measure; see [25] for details.

RPSLS. For the classical Janken (RPS) game, the set of hands is $\{\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3\}$ and all subsets of cardinality two are WOD sets, i.e., $\{\mathscr{H}_1, \mathscr{H}_2\}, \{\mathscr{H}_1, \mathscr{H}_3\}$ and $\{\mathscr{H}_2, \mathscr{H}_3\}$, all other subsets leading to ties. Thus **RPSLS** is an exp-game and Theorem 3 applies.

For instance, for $p_1 = p_2 = p_3 = \frac{1}{3}$, we have $\rho = \frac{2}{3}$ and $\nu = 3$ and one recovers the main result from [33]; see also Section 1.

Other three-hand games. Apart from **RPSLS** many other games with three hands are possible. We content ourselves here with a short discussion of games defined on a dominance graph with $p_1 = p_2 = p_3 = \frac{1}{3}$. All five connected dominance graphs are given as follows.



I II III IV V

The games based on these graphs are played in a natural way. Only the **GJLS** arising from Graph V requires some more explanation. Assume that the hands are $\{\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3\}$, where $\mathscr{H}_i$ corresponds to the i-th node from the top to the bottom on Graph V. Then all WOD sets are $\{\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3\}$ (with $\mathscr{H}_1$ the winning hand), $\{\mathscr{H}_1, \mathscr{H}_2\}$ (with $\mathscr{H}_1$ the winning hand) and $\{\mathscr{H}_2, \mathscr{H}_3\}$ (with $\mathscr{H}_2$ the winnings hand). In particular, note that we define $\{\mathscr{H}_1, \mathscr{H}_3\}$ as a tie (no transitivity); for a more general definition of games arising from general dominance graphs, see the paragraph on Janken games around the world below.

Note that the **GJLS** defined on Graph I corresponds to **RPSLS** and is the only exp-game; games on all other graphs are log-games.

For instance, the **GJLS** defined on Graph II has $\varpi_n = 1 - 3^{1-n}$, and (6) has the form

$$\mathbb{P}(I_n = j) = \begin{cases} 3^{1-n}, & \text{if } j = n; \\ 3^{-n}\binom{n}{j}\left(2^{n-j} + 1\right), & \text{if } 1 \leqslant j < n. \end{cases}$$

The mean and the variance are then described by applying Theorem 1. For the limit law, observe that (19) becomes

$$\tilde{A}_{\ell+1}(z) = \left(1 + e^{-z/3} + e^{-2z/3}\right)\tilde{A}_\ell\left(\frac{z}{3}\right).$$

Then the explicit expression in Remark 2 specializes to

$$\tilde{A}_\ell(z) = \frac{z}{3^\ell} e^{-z/3^\ell} \prod_{1 \leqslant j \leqslant \ell} \left(1 + e^{-z/3^j} + e^{-2z/3^j}\right),$$

and one then obtains for the sum on the right-hand side in Theorem 2

$$\begin{aligned} \sum_{j \geqslant 0} \tilde{R}_j(3^j z) &= z e^{-z} + z e^{-z} \sum_{k \geqslant 0} \left(e^{-3^k z} + e^{-2 \cdot 3^k z}\right) \prod_{0 \leqslant j < k} \left(1 + e^{-3^j z} + e^{-2 \cdot 3^j z}\right) \\ &= \frac{z}{e^z - 1}. \end{aligned}$$

Note that this gives the same result as for **CTLS** with an unbiased coin, with the only difference that 2 is replaced by 3 in (2); for a generalization with 2 replaced by any number m see the next paragraph below.

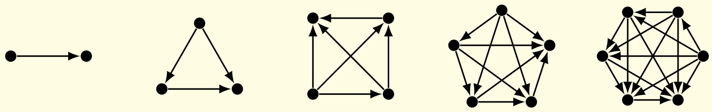
Consider now the games defined on Graphs III and IV, which can be seen to correspond to **CTLS** with a biased coin whose probability of head is either $\frac{1}{3}$ (Graph III) or $\frac{2}{3}$ (Graph IV). For example, for the game defined on Graph III, we have $\varpi_n = 1-(2^n+1)3^{-n}$ and then (6) becomes

$$\mathbb{P}(I_n = j) = \begin{cases} (2^n+1)3^{-n}, & \text{if } j = n; \\ \binom{n}{j}2^{n-j}3^{-n}, & \text{if } 1 \leqslant j < n. \end{cases}$$

Finally, the **GJLS** defined on Graph V is also equivalent to **CTLS** with a biased coin, and ϖ_n and (6) are identical to those of the **GJLS** on Graph III.

In summary, there are exactly five different three-hand games defined on connected dominance graphs, where hands are chosen uniformly at random. One of them is an exp-game and all others are log-games. Moreover, the exp-game is **RPSLS** and three of the four log-games correspond to **CTLS** with a biased coin. Finally, the fourth log-game is a natural extension of **CTLS** from two hands to three hands.

Janken games on acyclic cliques and CTLS. We consider Janken games defined on a directed acyclic clique. Note that such a clique contains exactly one node of out-degree i for $i = 0, \dots, m-1$; see the following graphs for $m = 2, \dots, 6$.



The **GJLS** (played in the natural way) on such a directed acyclic graph is a natural extension of **CTLS** with an unbiased coin, and the **GJLS** from the previous paragraph on Graph II. The functional equation for the moments becomes

$$\tilde{f}(z) = \left(1 + e^{-z/m} + e^{-2z/m} + \cdots + e^{-(m-1)z/m}\right)\tilde{f}\left(\frac{z}{m}\right) + \tilde{g}(z).$$

Thus the same normalization technique as that used for **CTLS** with an unbiased coin applies, and the Fourier coefficients in Theorem 1 can be made more explicit.

Moreover, for the asymptotic distribution, we have

$$\tilde{A}_{\ell+1}(z) = \left(1 + e^{-z/m} + e^{-2z/m} + \cdots + e^{-(m-1)z/m}\right)\tilde{A}_{\ell}\left(\frac{z}{m}\right),$$

which gives again the same form as (2) but with 2 there replaced by m.

Regular tournament Janken games. A *tournament* is a directed graph (digraph) obtained by assigning a direction for every edge in a complete graph. A *regular tournament* of $2m+1$ nodes is a tournament in which every node dominates exactly m other nodes and is dominated by the remaining m nodes.

Typical examples of regular tournaments are the dominance graphs of **RPSLS** and the rock-paper-scissors-Spock-lizard game. Also the dominance graph of the rock-paper-scissors-well game is a subtournament of the rock-paper-scissors-Spock-lizard game. Note that up to isomorphism there is only one regular tournament of 5 nodes, two regular tournaments of 7 nodes, and the number increases very rapidly with the number of nodes. The case of regular tournaments on infinitely many nodes also makes sense and will be briefly discussed in Section 5.

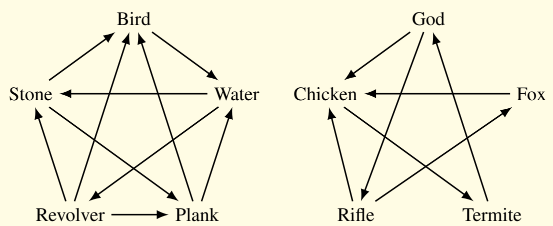
We consider a class of Janken games arising from a regular tournament generalizing the RPS game. The hands $\{\mathscr{H}_1, \ldots, \mathscr{H}_{2m+1}\}$ correspond to the nodes of the regular tournament. For instance, the hands can be ordered cyclically such that $\mathscr{H}_i$ dominates $\mathscr{H}_j$ for $j - i \equiv \{1, \ldots, m\} \bmod (2m+1)$. The game is played as follows: if the n players choose the hands $S = \{\mathscr{H}_{i_1}, \ldots, \mathscr{H}_{i_\ell}\}$, then the game is in a tie if and only if either of the following condition holds:

- S is a singleton, or
- the induced subgraph of S contains a cycle.

Otherwise the game is in a WOD situation.

Since by definition, the set of all hands is a tie, this is an exp-game. Thus the cost of this Janken game is described by Theorem 3. For instance, if $p_i = 1/(2m+1)$ for $1 \leqslant i \leqslant 2m+1$, then $\rho = (m+1)/(2m+1)$ and $\nu = 2m+1$. On the other hand, if we do not count rounds in which the induced subgraph of S contains a cycle, then the number of steps needed is predicted by Theorem 1 and Theorem 2; see Section 4 where counting without ties is considered in more generality.

Some other Janken games around the world. Many variants of RPS are played all over the world. Examples include (i) the rock-paper-scissors-well game in Germany; (ii) the bird-stone-revolver-plank-water game in Malaysia, and (iii) the god-chicken-rifle-termite-fox game in Guangdong province (in China); see the figure to the right for an illustration of (ii) and (iii). See also [33].



(MALAYSIA) (Guangdong, CHINA)

All the three games (i)–(iii) are mainly played as two-person games. They can be generalized to n-person games. More precisely, assume that the hands are given by $\{\mathscr{H}_1, \ldots, \mathscr{H}_m\}$. If the n players choose the hands $S = \{\mathscr{H}_{i_1}, \ldots, \mathscr{H}_{i_\ell}\}$, then we consider the subgraph induced by these hands. The game is in a tie if and only if either the subgraph consists only of isolated nodes or there is a cycle in one of the connected components of the subgraph. (Note that in contrast to the regular tournament Janken game, induced subgraphs are now not necessarily connected). Thus, if S is a WOD set (i.e., not a tie), then the induced subgraph has at least one node with only outgoing edges. All these nodes together with the isolated nodes form the set of winning hands; the remaining hands are the losing hands.

As an example, assume that in the god-chicken-rifle-termite-fox game, the players choose the hands $\{G, C, F\}$. Then this set is a WOD set where $\{G, F\}$ are the winning hands and $\{C\}$ is the losing hand.

The games (i)–(iii) are all of exponential type since there are cycles in the dominance graphs. Thus the number of rounds is described by Theorem 3. For instance, if hands are chosen uniformly at random, then one obtains the game indices listed in the right table.

Games on a circulant payoff matrix. These games where introduced in [47] for selecting a leader in a distributed system. In contrast to all the previous games, they are not defined via an underlying dominance graph. We first recall their definition.

GJLS	ρ	ν
(i) rock-paper-scissors-well	$\frac{3}{4}$	2
(ii) rock-stone-revolver-plank-water	$\frac{4}{5}$	1
(iii) god-chicken-rifle-termite-fox	$\frac{4}{5}$	3

The games are played with $2m+1$ hands $\{\mathscr{H}_1, \dots, \mathscr{H}_{2m+1}\}$, where for $1 \leqslant i, j \leqslant 2m+1$ with $i \neq j$, the hand $\mathscr{H}_i$ gets a payoff of $2^{m+g(i,j)}$ from $\mathscr{H}_j$. Here $i-j \equiv g(i,j) \bmod (2m+1)$ with $-m \leqslant g(i,j) \leqslant m$. If the players choose the hands $S = \{\mathscr{H}_{i_1}, \dots, \mathscr{H}_{i_\ell}\}$, then the total gain of $\mathscr{H}_{i_j}$ is defined as $G_{i_j,S} = \sum_{\mathscr{H}_{i_k} \in S} 2^{m+g(i_j,i_k)}$. Denote by S_M the subset of hands from S with the maximal total gain. Then S is a WOD set with winning hands S_M and losing hands $S \setminus S_M$ if and only if $\#S_M < \#S$.

Examples of a three-hand game and a five-hand game defined on circulant payoff matrices are given in Figure 1. Note that $m = 1$ corresponds to **RPSLS**. On the other hand, the game with $m = 2$ is different from the rock-paper-scissors-Spock-lizard game since there are more WOD sets in the former. This is also the reason why the five-hand circulant payoff matrix game was described as more efficient than the five-hand regular tournament Janken game in [47].

hands	rock	paper	scissors
rock	2^1	2^0	2^2
paper	2^2	2^1	2^0
scissors	2^0	2^2	2^1

hands	$\mathscr{H}_1$	$\mathscr{H}_2$	$\mathscr{H}_3$	$\mathscr{H}_4$	$\mathscr{H}_5$
$\mathscr{H}_1$	2^2	2^1	2^0	2^4	2^3
$\mathscr{H}_2$	2^3	2^2	2^1	2^0	2^4
$\mathscr{H}_3$	2^4	2^3	2^2	2^1	2^0
$\mathscr{H}_4$	2^0	2^4	2^3	2^2	2^1
$\mathscr{H}_5$	2^1	2^0	2^4	2^3	2^2

Table 1: *Three-hand (left) and five-hand (right) circulant payoff matrix game.*

The set of all hands in a circulant payoff matrix game is obviously a tie because each hand has the same total gain. Thus, circulant payoff matrix games are of exponential type and their complexity are described by Theorem 3. For instance, if $p_i = 1/(2m+1)$ for $1 \leqslant i \leqslant 2m+1$, then $\rho = 2m/(2m+1)$ and $\nu = 2m+1$. This generalizes the observation made in [47], namely, *$(2m+1)$-hand circulant payoff matrix games are more efficient than $(2m+1)$-hand regular tournament games.*

4 Extensions and Other Gap Theorems

Another equally important cost measure for **GJLS** is the total number of hands used until a leader is selected, which corresponds to the number of times the random variate is generated if the hands are generated by random mechanisms. We show that instead of the log-exp complexity change, there is a change from linear to exponential for the expected cost; also the limit law exists in either case (log or exp).

On the other hand, we saw, from the above analysis, that ties play a crucial role in distinguishing between log-games and exp-games. We examine in this section the contribution of ties in slightly more detail. More precisely, we discuss the number of rounds until a leader is selected when ties are ignored.

4.1 The total number of hands

We consider here the total number of hands Y_n used by all players before finding a leader by **GJLS**. It turns out that Y_n exhibits a scale change from linear ($\rho = 1$) to exponential ($\rho < 1$).

We begin with the case $\rho = 1$ for which we have the recurrence (cf. (5))

$$Y_n \stackrel{d}{=} Y_{I_n} + n, \qquad (n \geqslant 2),$$

where Y_n, I_n are independent, $Y_1 = 0$ and I_n is defined in (6).

While the asymptotic distribution of X_n is dictated by periodic oscillations, the distribution of Y_n follows a central limit theorem.

Theorem 4 (Log-games)**.** *The total number of hands Y_n used by **GJLS** to find a leader in log-games ($\rho = 1$) is asymptotically normally distributed*

$$\frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\mathbb{V}(Y_n)}} \xrightarrow{d} \mathscr{N}(0,1),$$

where $\mathscr{N}(0,1)$ denotes the standard normal distribution. Furthermore, the mean and the variance satisfy

$$\begin{cases} \mathbb{E}(Y_n) = \frac{1}{1-\alpha} n + P_3(\log_{1/\alpha} n) + o(1), \\ \mathbb{V}(Y_n) = \frac{\alpha}{(1-\alpha)^2} n + P_4(\log_{1/\alpha} n) \log n + O(1), \end{cases} \tag{28}$$

where $P_3(z), P_4(z)$ are one-periodic functions.

Proof. (Sketch) As in Section 2.2, we consider

$$\tilde{f}_1(z) := e^{-z} \sum_{n \geqslant 1} \mathbb{E}(Y_n) \frac{z^n}{n!} \qquad \text{and} \qquad \tilde{f}_2(z) := e^{-z} \sum_{n \geqslant 1} \mathbb{E}(Y_n^2) \frac{z^n}{n!},$$

which, by a similar computation as in Section 2.2, satisfy the functional equations

$$\begin{aligned} \tilde{f}_1(z) &= \tilde{f}_1(\alpha z) + z + \tilde{\phi}_1(z) \\ \tilde{f}_2(z) &= \tilde{f}_2(\alpha z) + 2z \tilde{f}_1(\alpha z) + 2\alpha z \tilde{f}_1'(\alpha z) + z^2 + z + \tilde{\phi}_2(z), \end{aligned}$$

where $\tilde{\phi}_1(z)$ and $\tilde{\phi}_2(z)$ are finite sums of exponentially small terms.

The proof of (28) then follows the same line of arguments as that of Theorem 1 except for the variance for which we need to consider the Poissonized variance (see [20])

$$\tilde{V}(z) := \tilde{f}_2(z) - \tilde{f}_1(z)^2 - z \tilde{f}_1'(z)^2,$$

which then satisfies the equation

$$\tilde{V}(z) = \tilde{V}(\alpha z) + \alpha(1-\alpha) z \tilde{f}_1'(\alpha z)^2 + \tilde{\phi}_3(z),$$

where $\tilde{\phi}_3(z)$ consists of exponentially small terms.

Finally, the central limit theorem can be proved either by the method of moments using the shifting-the-mean technique (see [5]) or by the contraction method [35]; details are omitted here. ∎

Note that for **CTLS** with an unbiased coin, it was proved in [39] that $\mathbb{E}(Y_n) = 2n$ ($\alpha = \frac{1}{2}$). This is a rather exceptional result which does not seem to hold in general. Furthermore, in this case, $P_4(z) \equiv 0$.

We now consider exp-games ($\rho < 1$). Here the total number of hands used satisfies the distributional recurrence

$$Y_n \stackrel{d}{=} Y_{J_n} + nT_n, \qquad (n \geqslant 2),$$

where Y_n, J_n, T_n are independent, T_n is a geometrically distributed random variable with parameter ϖ_n, $Y_1 = 0$ and J_n is defined in (9).

Then the same approach used in Section 2.3 applies and we obtain the same exponential limit law for Y_n.

Theorem 5 (Exp-games)**.** *The total number of hands Y_n used to find a leader by* ***GJLS*** *in exp-games ($\rho < 1$) satisfies*

$$\frac{\nu\rho^n}{n} Y_n \stackrel{d}{\longrightarrow} \mathrm{Exp}(1),$$

where Exp(1) *denotes an exponentially distributed random variable with mean one. In addition, we have convergence of all moments. In particular, the mean satisfies*

$$\mathbb{E}(Y_n) \sim \frac{n}{\nu\rho^n}.$$

Overall, we see that the total number of hands used by **GJLS** exhibits a sharp scale change from linear to exponential.

4.2 Counting without ties

Denote by Z_n the number of rounds used by **GJLS** until a leader is selected where ties are ignored. Then Z_n satisfies the same recurrence as (8) with T_n there replaced by 1, namely,

$$Z_n \stackrel{d}{=} Z_{J_n} + 1, \qquad (n \geqslant 2),$$

where Z_n, J_n are independent, $Z_1 = 0$ and J_n is defined in (9). It turns out that in such a case the expected cost is always logarithmic, independent of ρ. For simplicity, we consider only the mean.

We introduce first some notations. Denote by $S_1, \dots S_\nu$ the WOD sets for which the maximum is attained in (4). Furthermore, define $\alpha_\ell := \sum_{\mathscr{H}_j \in W_\ell} p_j$ for $1 \leqslant \ell \leqslant \nu$, where W_ℓ is the set of winning hands belonging to S_ℓ.

By the same arguments used in Section 2.2, we obtain the functional equation

$$\tilde{f}_1(z) = \frac{1}{\nu} \sum_{1\leqslant\ell\leqslant\nu} \tilde{f}_1\Big(\frac{\alpha_\ell}{\rho} z\Big) + 1 + \tilde{\phi}_1(z),$$

with $\tilde{f}_1(0) = \tilde{\phi}_1(0) = 0$, where $\tilde{f}_1(z) := e^{-z} \sum_{n\geqslant 1} \mathbb{E}(Z_n) \frac{z^n}{n!}$, and $\tilde{\phi}_1(z)$ is a finite sum of exponentially small terms.

Asymptotics of $\mathbb{E}(Z_n)$ then follows from the same Mellin and de-Poissonization arguments we used above. The major difference here is that the Mellin transform of $\tilde{f}_1(z)$ has now the denominator $1 - \frac{1}{\nu} \sum_{1\leqslant\ell\leqslant\nu} \big(\frac{\rho}{\alpha_\ell}\big)^s$, instead of $1 - \alpha^{-s}$ when $\rho = 1$; see (18).

For the inverse Mellin transform, one needs to clarify the set of zeros of this function for which much has been known; see [9, 12] for more information. First, it is easy to see that all zeros must satisfy $\Re(s) > 0$. Moreover, it is well-known that $s = 0$ is the only zero on the vertical line $\Re(s) = 0$ if and only if at least one of the ratios $\log(\rho/\alpha_j)/\log(\rho/\alpha_k)$ is irrational. Note that if the latter does not hold, then there exits an $r > 1$ such that $\rho/\alpha_j = r^{m_j}$ for positive integers m_j, and there are infinitely many zeros on $\Re(s) = 0$ that are equally spaced along this line; for these and related properties see, e.g., [12] and references therein.

Using this and the approach from Section 2.2, we obtain the following result.

Theorem 6. *The expected number of rounds Z_n, counted without ties, used by* **GJLS** *to find a leader satisfies*

$$\mathbb{E}(Z_n) = h_\nu \log n + P(\log_r n) + o(1),$$

where $h_\nu := \frac{1}{\frac{1}{\nu}\sum_{1\leqslant \ell\leqslant \nu}\log(\rho/\alpha_\ell)}$, and $P(z)$ is a constant if at least one of the ratios $\frac{\log(\rho/\alpha_j)}{\log(\rho/\alpha_k)}$ is irrational, and is a one-periodic function otherwise.

Thus ties dominate in an exp-game.

5 A concluding remark

In this paper, we discussed leader selection based on generalized Janken games. Our framework contains as special cases the classical leader selection procedure using coin-tossing, which is widely used in computer science and related areas, and the popular two-person RPS game (and its variants), which is played in many countries and is of importance in game theory, biology and physics. We showed that leader selection based on the latter as well as many other previous examples of Janken games exhibit only dichotomous behaviors: either very efficient of log-order or very laborious of exp-order.

We conclude this paper with a remark on an infinite-hand Janken game generalizing the regular tournament Janken game introduced in Section 3.

Assume that the n players choose points uniformly at random on the surface of the unit sphere in the ℓ-space. Then the probability that all points lie on the same hemisphere [52] is

$$2^{1-n}\sum_{0\leqslant j<\ell}\binom{n-1}{j}.$$

Consider the unit circle ($\ell = 2$). Assume that two players choose points P and Q, respectively. We define the dominance relation according to the clockwise arc-length: if the clockwise arc-length from Q to P is larger than the counter-clockwise one, then we say that Q dominates P (or Q wins); see [24]. Obviously, there is a winner whenever all players choose points lying on the same semicircle; otherwise, the game is in a tie. The expected time until a winner is selected is $1/p - 1$, where p is the probability that all points lie on the same semicircle. By the above result, the latter probability is given by $p = n/2^{n-1}$ and, consequently, the expected time is given by $n^{-1}2^{n-1} - 1$. This is again a game of exponential type. Note that its discrete version is the regular tournament Janken game with $p_i = 1/(2m+1)$ for $1 \leqslant i \leqslant m$.

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